MINIMAL BRATTELI DIAGRAMS AND DIMENSION GROUPS OF AF C*-ALGEBRAS

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ABSTRACT. A method is described which identifies a wide variety of AF algebra dimension groups with groups of continuous functions. Since the continuous functions in these groups have domains which correspond to the set of all infinite paths in what will be called minimal Bratteli diagrams, it becomes possible, in some cases, to analyze the dimension group's order preserving automorphisms by utilizing the topological structure of the associated minimal Bratteli diagram.

1. Introduction

In studying approximately finite-dimensional (AF) C*-algebras $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$, Bratteli, in his seminal paper [5], introduced a certain infinite graph, now called a Bratteli diagram, which can be used to encode the nature of the subalgebras $\mathfrak{A}_n$ and the actions of the connecting homomorphisms ϕ_n. These diagrams, although not unique for a given algebra, can be used to ascertain certain characteristics of the algebra, such as in [5] and [13].

Subsequently, Elliott [9] proved his theorem showing that dimension groups provide a complete isomorphism invariant for AF algebras. These dimension groups can be realized as a direct limit of a sequence of scaled ordered groups. In this context, the Bratteli diagram is useful since the exact nature of the groups in this sequence and of the connecting maps between them can be readily obtained from it.

In the case that the AF algebra under consideration is commutative, another interesting way in which the Bratteli diagram plays a role in the description of the dimension group can be observed. To be specific, let X be a compact metric space with a basis consisting of sets which are simultaneously open and closed (clopen). Then $C(X)$ is AF and the associated dimension group can be shown to be order isomorphic to the scaled ordered group $(C(X,\mathbb{Z}), C(X,\mathbb{Z}^+), \chi_X)$ where $C(X,\mathbb{Z})$ are the continuous functions $f : X \to \mathbb{Z}$, $C(X,\mathbb{Z}^+)$ are those functions

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in $C(X,\mathbb{Z})$ which only take nonnegative integer values, and χ_X is the function which is always 1.

For commutative AF algebras such as this, the spectrum X can be identified with the set of all infinite paths in the associated Bratteli diagram. Therefore, in such a case, the dimension group can be conveniently described as a set of integer-valued continuous functions with domain equal to the Bratteli diagram. This particularly simple description of the dimension group of $C(X)$ is one source of motivation for the results of this paper.

From the viewpoint of K-theory, cf. [1], the dimension group of an AF algebra $\mathfrak{A}$ turns out to be the K_0 group of $\mathfrak{A}$, $K_0(\mathfrak{A})$. We note here that ordered groups of the form $C(X,\mathbb{Z})$, where X is a Cantor set, have also made appearances in the K-theoretic calculations of a number of authors. These have included [8, 10, 11, 16], where crossed product C*-algebras were studied, and a determination of their K-theory utilized the fact that $K_0(C(X))$ is isomorphic to $C(X,\mathbb{Z})$.

In this paper a type of Bratteli diagram, which we refer to as a *minimal Bratteli diagram* (definition below), will play an important role. In the case of the AF algebra $C(X)$ above, the Bratteli diagram is already minimal, and we see that $K_0(C(X))$ can be realized as $C(X,\mathbb{Z})$. More generally, we will consider certain AF algebras $\mathfrak{A}$ which have proper minimal diagrams. That is, the Bratteli diagram itself may not be minimal, but a subgraph is. Then, by defining X_{min} to be the set of all infinite paths in the minimal diagram, it will be possible to describe $K_0(\mathfrak{A})$ as a subgroup of $C(X_{min},G)$, where now G is in general a subset of $\mathbb{Q}$, and under certain hypotheses, $K_0(\mathfrak{A})\cong C(X_{min},G)$. In this sense, the results of this paper provide a direct generalization of the observation that $K_0(C(X))\cong C(X,\mathbb{Z})$.

For an arbitrary AF algebra, there may correspond multiple, non-homeomorphic versions of X_{min}. However, in Section 5, we are able to show that when $\mathfrak{A}$ has a diagram with a unique graph corresponding to the set X_{min}, then the topological structure of X_{min} provides a way to discriminate between non-isomorphic algebras (Corollary 5.1). This result is a generalization and provides a new proof of the well known fact that isomorphic commutative AF algebras have homeomorphic spectra. These facts rely on Theorem 4.4, which can also be used to provide information about the automorphism group of certain dimension groups in terms of the homeomorphisms on the set X_{min}. In particular, those dimension groups to which this applies will be those arising from AF algebras that have Bratteli diagrams with a unique choice for the graph which corresponds to the set X_{min}. It should be

noted that these AF algebras are, in general, different than the AF algebras with stationary Bratteli diagrams considered in [2, 3, 4]. There, questions about stable isomorphisms of these AF algebras and the order preserving isomorphisms of their corresponding dimension groups are considered.

In addition to being useful from a theoretical perspective, the results contained here provide an algorithm for determining the K-theory of many AF algebras, with the resulting dimension groups being groups of continuous functions. The types of AF algebras for which the results of this paper apply include the UHF and GICAR algebras, with some of the results here being a generalization of the calculations done for the GICAR algebra in [14]. The results of [14] are in the context of inverse semigroups of partial homeomorphisms, and it is this dynamical systems perspective on AF algebras which has, to some extent, motivated the present study. Further recent work in this area is that of [7], which considers the construction of AF algebras (among others) by utilizing partial actions. As pointed out there, there is no substantive difference between the partial action approach and the partial homeomorphism approach. As such, the results here might possibly be generalized to fruitfully study other types of algebras.

2. Preliminaries

Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$ be an AF C*-algebra with finite-dimensional subalgebras $\mathfrak{A}_n \cong M_{k(1,n)} \oplus \cdots \oplus M_{k(m_n,n)}$, for all $n \geq 0$ (with $\mathfrak{A}_0 \cong \mathbb{C}$), and suppose that for each n, the connecting maps $\phi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ are unital injective $*$-homomorphisms. Let $\mathfrak{D} \subset \mathfrak{A}$ be the abelian subalgebra $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \phi_n)$, where $\mathfrak{D}_n$ is the subalgebra of $\mathfrak{A}_n$ spanned by the diagonal matrix units in $\mathfrak{A}_n$, and let $X_{\mathfrak{A}}$ be the Gelfand spectrum of $\mathfrak{D}$. Each element of $X_{\mathfrak{A}}$ corresponds uniquely to a decreasing sequence of diagonal matrix units in $\mathfrak{A}$. To be more specific, for any $n \geq 0$, $1 \leq r \leq m_n$, and $1 \leq s \leq k(r,n)$, let $e_{s,s}(r,n) = e_s(r,n) \in M_{k(r,n)}$ be a diagonal matrix unit of $\mathfrak{A}$. Up to unitary equivalence each mapping ϕ_n is a standard embedding in the sense of [15]. Therefore, $\phi_n(e_s(r,n))$ is a sum of diagonal matrix units in $\mathfrak{A}_{n+1}$, say

$$\phi_n(e_s(r,n)) = \sum_{l=1}^{c} e_{\beta_l}(\alpha_l, n+1),$$

where c depends on the embedding ϕ_n and the matrix unit $e_s(r,n)$. The elements of $X_{\mathfrak{A}}$ are such that if $e_{s_0}(r_0,0) \geq e_{s_1}(r_1,1) \geq \ldots$ is one of the sequences in $X_{\mathfrak{A}}$ then $e_{s_{n+1}}(r_{n+1}, n+1)$ appears in the sum $\phi_n(e_{s_n}(r_n,n))$, for all $n \geq 0$.

We now define the set $\widehat{e_s}(r,n) \subset X_{\mathfrak{A}}$ to be the collection of all sequences in $X_{\mathfrak{A}}$ with the matrix unit $e_s(r,n)$ as the n-th coordinate. Then, the topology on $X_{\mathfrak{A}}$ has as a basis the collection

$$\bigvee_{n=1}^{\infty}\bigvee_{r=1}^{m_n}\bigvee_{s=1}^{k(r,n)} \widehat{e_s}(r,n).$$

Of course, from the Gelfand theory, $X_{\mathfrak{A}}$ is compact Hausdorff, and in fact, each basis element $\widehat{e_s}(r,n)$ is clopen. Therefore, $X_{\mathfrak{A}}$ is a 0-dimensional compact Hausdorff space.

For the purposes of this paper, a certain closed subset of $X_{\mathfrak{A}}$ will play an important role. We first define, for each $n \geq 0$, the set $X_{min}^n = \bigcup_{r=1}^{m_n} \widehat{e}_1(r,n)$, and let the subset $X_{min} \subset X_{\mathfrak{A}}$, with the topology it inherits from $X_{\mathfrak{A}}$, be given by

$$X_{min} = \bigcap_{n=0}^{\infty} X_{min}^n.$$

We will prove the following fact about the set X_{min}.

Lemma 2.1. *The sequence $\{X_{min}^n\}_{n=0}^{\infty}$ is a nested decreasing sequence of clopen subsets of $X_{\mathfrak{A}}$, and therefore X_{min} is a nonempty closed subset of $X_{\mathfrak{A}}$.*

Proof. For $n \geq 0$ given, let $e_s(r,n)$ be a diagonal matrix unit in $\mathfrak{A}_n$ where $s > 1$ and suppose, to derive a contradiction, that

$$\phi_n(e_s(r,n)) = \sum_{l=1}^{c} e_{\beta_l}(\alpha_l, n+1)$$

where $\beta_l = 1$ for some l. Since (up to unitary equivalence) ϕ_n is a standard embedding, it maps the strictly upper triangular subalgebra of $\mathfrak{A}_n$ to the strictly upper triangular subalgebra of $\mathfrak{A}_{n+1}$. Therefore, $\phi_n(e_{s-1,s}(r,n))$ will be a sum of the form

$$\phi_n(e_{s-1,s}(r,n)) = \sum_{l=1}^{c} e_{\gamma_l,\delta_l}(\alpha_l, n+1)$$

where $\gamma_l < \delta_l$, for all l. The fact that ϕ_n is a $*$-homomorphism means

$$\begin{aligned}
\phi_n(e_s(r,n)) &= \phi_n(e_{s-1,s}(r,n))^*\phi_n(e_{s-1,s}(r,n)) \\
&= \sum_{l=1}^{c}\sum_{l'=1}^{c} e_{\delta_l,\gamma_l}(\alpha_l, n+1)e_{\gamma_{l'},\delta_{l'}}(\alpha_{l'}, n+1).
\end{aligned}$$

Because ϕ_n maps the diagonal of $\mathfrak{A}_n$ to the diagonal of $\mathfrak{A}_{n+1}$, if $\gamma_l = \gamma_{l'}$ then $\delta_l = \delta_{l'}$. But this implies that $l = l'$. So

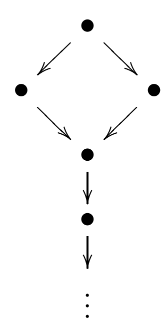
$$\phi_n(e_s(r,n)) = \sum_{l=1}^{c} e_{\delta_l}(\alpha_l, n+1).$$

By assumption, it follows that $\delta_{\bar{l}} = 1$ for some $\bar{l}$. Since $\gamma_l < \delta_l$ for all l, we have that $\gamma_{\bar{l}} < 1$, a contradiction. Hence, if $s > 1$ then $\phi_n(e_s(r,n)) = \sum_{l=1}^{c} e_{\beta_l}(\alpha_l, n+1)$ where $\beta_l > 1$ for all l.

Now, take $(x_1, x_2, \ldots) \in \widehat{e_1}(r, n+1) \subset X_{min}^{n+1}$. By the contrapositive of the fact just proven, it must be that $(x_1, x_2, \ldots) \in \widehat{e_1}(\overline{r}, n)$ for some $\overline{r}$. So, $X_{min}^{n+1} \subset X_{min}^{n}$, for all $n \geq 0$. Hence, $\{X_{min}^{n}\}_{n=0}^{\infty}$ is a nested, decreasing sequence of sets. Furthermore, since $\widehat{e_1}(r,n)$ is clopen for all n and all r, so too is X_{min}^{n} clopen, for all $n \geq 0$. It then follows that X_{min} is closed, and by the compactness of $X_{\mathfrak{A}}$ that we have $\bigcap_{n=0}^{\infty} X_{min}^{n} \neq \emptyset$. $\square$

At this point we make the observation that for a given AF algebra $\mathfrak{A}$, the set X_{min} depends on the exact nature of the standard embeddings ϕ_n.

Example 2.2. Consider the AF algebra $\mathfrak{A}$ with Bratteli diagram



Here, we have

$$X_{\mathfrak{A}} = \{(e_1(1,0), e_1(1,1), \ldots), (e_1(1,0), e_1(2,1), \ldots)\}.$$

If we suppose $\phi_1 : \mathfrak{A}_1 \cong \mathbb{C} \oplus \mathbb{C} \to \mathfrak{A}_2 \cong M_2$ is such that

$$\phi_1(a \oplus b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

then $X_{min} = \{(e_1(1,0), e_1(1,1), \ldots)\}$. However, by assuming

$$\phi_1(a \oplus b) = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix},$$

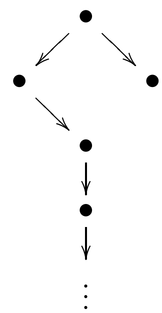
we arrive at $X_{min} = \{(e_1(1,0), e_1(2,1), \ldots)\}$.

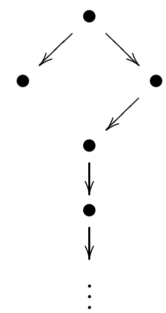
This example is a simple one meant to illustrate the fact that by changing the sequence of standard embeddings $\{\phi_n\}_{n=0}^{\infty}$ to a sequence of unitarily equivalent standard embeddings, we may arrive at a different set X_{min}. Of course, in this example, these different copies of X_{min} are homeomorphic. So, in this sense, Example 2.2 does not seem very revealing. However, we will see in a moment (Example 2.3) that by making different, unitarily equivalent, choices for the sequence of standard embeddings, non-homeomorphic copies of X_{min} can arise from the same diagram.

For a given AF algebra $\mathfrak{A}$, the Bratteli diagram provides a convenient tool for visualizing these different possibilities for the set X_{min}. The basis for this begins with the fact that the set $X_{\mathfrak{A}}$ can be realized as the set of all infinite paths in the Bratteli diagram for $\mathfrak{A}$.

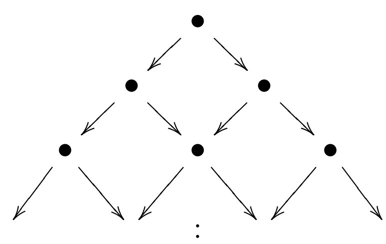
To establish some notation for the Bratteli diagram of $\mathfrak{A}$, we label the vertices at level n by $k(1,n),\ldots,k(m_n,n)$, analogous to the labelling scheme for the summands which constitute $\mathfrak{A}_n$. Now, for each $n \geq 1$, delete enough edges between the vertices $k(1,n-1),\ldots,k(m_{n-1},n-1)$, and the vertices $k(1,n),\ldots,k(m_n,n)$, so that each vertex $k(r,n)$, $1 \leq r \leq m_n$, absorbs exactly one edge from the level above. The set of all infinite paths in the resulting subgraph of the Bratteli diagram will correspond to the set X_{min} for one possible choice of the standard embeddings $\{\phi_n\}_{n=0}^{\infty}$. To see that this is true, we first note that for all $n \geq 0$, each vertex $k(i,n)$, $1 \leq i \leq m_n$, can be thought of as corresponding to an upper left diagonal matrix unit of $\mathfrak{A}_n$. When viewed from this perspective, the proof of Lemma 2.1 shows us that there is a mapping from the vertices $\{k(1,n),\ldots,k(m_n,n)\}$ to the vertices $\{k(1,n-1),\ldots,k(m_{n-1},n-1)\}$, for all $n \geq 1$. Furthermore, this mapping can be represented by deleting all but m_n edges between the vertices $\{k(i,n-1) : 1 \leq i \leq m_{n-1}\}$ and $\{k(j,n) : 1 \leq j \leq m_n\}$ in the Bratteli diagram so that each vertex $k(j,n)$ absorbs exactly one edge. The resulting subgraph of the Bratteli diagram will then encode, via the set of all infinite paths, all the possible sequences of the form $(e_1(i_0,0), e_1(i_1,1),\ldots)$ in $X_{\mathfrak{A}}$, which is exactly the set X_{min}. By making different choices for the way we delete the edges, one is in effect making different, albeit unitarily equivalent, choices for the sequence of standard embeddings. In the context of Example 2.2, the two possible

diagrams representing X_{min} would therefore be

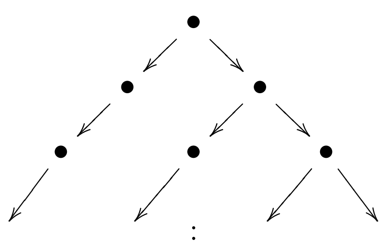




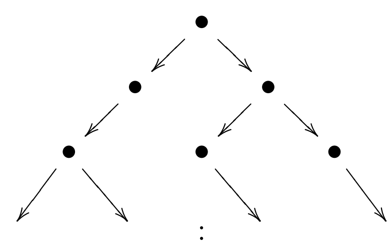
Example 2.3. To see that non-homeomorphic copies of X_{min} are possible for a given Bratteli diagram, consider the GICAR algebra, with Bratteli diagram



Two possible diagrams representing X_{min}, achieved through edge deletions as described above, are



and



The sets of all infinite paths in these diagrams are homeomorphic to

$$\overline{\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}} \qquad \text{and} \qquad \overline{\left\{\pm\left(1-\frac{1}{n}\right) : n \in \mathbb{Z}^+\right\}},$$

respectively. Since the former has one non-isolated point and the latter two such points, they are clearly not homeomorphic.

3. Minimal Bratteli Diagrams

Given an AF algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$, with, for all $n \geq 0$, $\mathfrak{A}_n \cong M_{k(1,n)} \oplus \cdots \oplus M_{k(m_n,n)}$, consider the sequence $\{m_n\}_{n=0}^{\infty}$ of positive integers. Two possibilities exist:

(1) $\limsup_{n\to\infty} m_n = L < \infty$; or
(2) $\limsup_{n\to\infty} m_n = \infty$.

For case (1), there exists a subsequence $\{m_{n_k}\}_{k=1}^{\infty}$ of $\{m_n\}_{n=0}^{\infty}$ such that $m_{n_k} = L$, for all $k \geq 1$, and for case (2), there exists a subsequence $\{m_{n_k}\}_{k=1}^{\infty}$ such that $m_{n_k} < m_{n_{k+1}}$, for all $k \geq 1$. For either situation, the sequence $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ of finite-dimensional subalgebras can be *contracted* to the subsequence $\{\mathfrak{A}_{n_k}\}_{k=0}^{\infty}$ (where $n_0 = 0$) with

$$\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n) = \varinjlim(\mathfrak{A}_{n_k}, \phi_{n_k}).$$

Therefore, without a loss of generality, we may assume that every AF algebra $\mathfrak{A}$ is such that either $\{m_n\}_{n=0}^{\infty}$ is a constant sequence or it is strictly monotonically increasing.

Remark 1. There may be many different ways to contract (*telescope* in the terminology of [10]) a given sequence $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ such that the above is true of $\{m_n\}_{n=0}^{\infty}$. In fact, it may be possible to find multiple contractions of both types.

As described by [6] and [15], each connecting homomorphism $\phi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$, for an AF algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$, has associated with it a *multiplicity matrix* $\overline{A}_{n,n+1} \in M_{m_{n+1},m_n}(\mathbb{N})$ which, up to unitary equivalence, completely describes the action of the mapping. Assuming that the sequence $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ has been contracted so as to fit into one of cases (1) or (2), we will, for the purposes of this paper, *assume that each of the multiplicity matrices* $\overline{A}_{n,n+1}$ *has full rank*. That is, for all $n \geq 0$, we will assume $\operatorname{rank}(\overline{A}_{n,n+1}) = m_n$. So, we are interested in those AF algebras for which there exists a contraction of either type (1) or (2) and for which this contraction results in all multiplicity matrices being of full rank.

For the moment, we focus our attention on case (2), where $m_n < m_{n+1}$, for all $n \geq 0$. We want to show that this case can be reduced to the situation where $m_n = n + 1$, for all $n \geq 0$.

Suppose that for some $n \geq 0$ we have $m_n + 1 < m_{n+1}$. Since we are assuming $\operatorname{rank}(\overline{A}_{n,n+1}) = m_n$, there are m_n linearly independent rows in $\overline{A}_{n,n+1}$. Let $P \in M_{m_{n+1}}$ be the permutation matrix which permutes the rows of $\overline{A}_{n,n+1}$ in such a way that $P\overline{A}_{n,n+1} = [b_{ij}] \in M_{m_{n+1},m_n}(\mathbb{N})$ has m_n linearly independent rows of $\overline{A}_{n,n+1}$ as its last m_n rows. Then,

the matrices

$$B_1 = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m_n} \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{2,1} & \cdots & b_{2,m_n} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \ldots,$$

$$B_{m_{n+1}-m_n-1} = \begin{bmatrix} 1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{m_{n+1}-m_n-1,1} & \cdots & b_{m_{n+1}-m_n-1,m_n} \\ 0 & \cdots & 0 & 1 & & \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{bmatrix},$$

and

$$B_{m_{n+1}-m_n} = \begin{bmatrix} 1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{m_{n+1}-m_n,1} & \cdots & b_{m_{n+1}-m_n,m_n} \\ 0 & \cdots & 0 & b_{m_{n+1}-m_n+1,1} & \cdots & b_{m_{n+1}-m_n+1,m_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{m_{n+1},1} & \cdots & b_{m_{n+1},m_n} \end{bmatrix}$$

are such that

$$P\overline{A}_{n,n+1} = B_{m_{n+1}-m_n} B_{m_{n+1}-m_n-1} \cdots B_2 B_1,$$

and each matrix B_i, $1 \leq i \leq m_{n+1} - m_n$, has full rank. Furthermore, each matrix $B_1, \ldots, B_{m_{n+1}-m_n-1}, P^{-1}B_{m_{n+1}-m_n}$ can be viewed as a multiplicity matrix for some unital embedding between finite-dimensional C*-algebras. We can therefore *dilate* (*microscope* in the terminology of [10]) the sequence $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ to a sequence $\{\mathfrak{B}_n\}_{n=0}^{\infty}$ of finite-dimensional C*-algebras such that $\{\mathfrak{B}_n\}_{n=0}^{\infty}$ can be contracted to a subsequence equal to $\{\mathfrak{A}_n\}_{n=0}^{\infty}$. Hence, we would have

$$\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n) = \varinjlim(\mathfrak{B}_n, \phi'_n),$$

with the additional properties that the sequence $\{\mathfrak{B}_n\}_{n=0}^{\infty}$ is such that $m_n = n + 1$, for all $n \geq 0$, and the multiplicity matrices corresponding

to the connecting embeddings all have full rank. We therefore assume, without a loss of generality, that for case (2), our AF algebras $\mathfrak{A}$ are such that $m_n = n+1$, for all $n \geq 0$.

As mentioned in Section 2, for a given AF algebra, many different possibilities may exist for the set X_{min}. For the proof of this paper's main result, it will be necessary for X_{min} to be chosen so that it is, in some sense, large. Our assumption that each multiplicity matrix has full rank is sufficient to guarantee such a choice always exists, and the remainder of this section will be devoted to a consideration of precisely what we mean by the word "large".

To begin, we consider an AF algebra of type (2). That is, $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$ where $m_n = n+1$ and $\operatorname{rank}(\overline{A}_{n,n+1}) = n+1$, for all $n \geq 0$. We will employ the method described in Section 2 of deleting certain edges between the vertices in each level of the Bratteli diagram of $\mathfrak{A}$ in order to describe X_{min} as a set of infinite paths in the resulting subgraph.

For the moment, we adopt a general perspective. Let $n \geq 1$ be given. We will consider graphs with the following properties:

Properties 3.1.

(I) There are $2n+1$ vertices which are arranged so that n vertices appear in one horizontal row (which we will refer to as level 1) and the remaining $n+1$ vertices appear in a horizontal row below the first (which we will refer to as level 2).

(II) The only edges are those connecting vertices at different levels. In other words, no vertices at the same level are connected by an edge.

(III) Every vertex is connected to another by at least on edge.

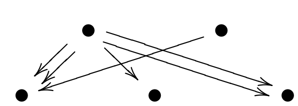
We label the set of all such graphs G_n.

Remark 2. The AF algebras with $m_n = n+1$, for all $n \geq 0$, are built from graphs of this form.

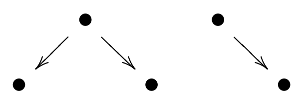
Definition 3.2. Given a graph $\Gamma \in G_n$, we will call the graph γ a *reduction of* Γ if γ is a subgraph of Γ obtained by deleting only edges and $\gamma \in G_n$. A graph in G_n will be called *minimal* if it has $n+1$ edges.

Remark 3. Any graph in G_n must have at least $n+1$ edges by Properties 3.1 (III). Therefore, no nontrivial reductions of minimal graphs exist.

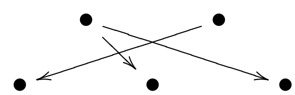
Example 3.3. It is easy to see that minimal reductions are not unique. The graph



is an element of G_2, with both

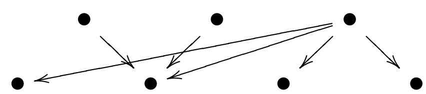


and



being minimal reductions.

Example 3.4. For an arbitrary n, not every element of G_n has a minimal reduction. The following element of G_3 is such that every way of deleting all but four edges results in an element not in G_3.



For our purposes, we will be interested in Bratteli diagrams which, at each level, have minimal reductions. Example 3.4 demonstrates that not all Bratteli diagrams will have minimal reductions at each level. However, if a Bratteli diagram does have a minimal reduction at each level, then by carrying out this reduction for all levels in the diagram, the corresponding "reduction" of the Bratteli diagram will correspond to one possible choice for X_{min} and will itself be a Bratteli diagram. One might call such a graph a *minimal Bratteli diagram* since deleting any more edges will create a subgraph which is no longer a Bratteli diagram. This will be an important object for our main result. We are therefore interested in answering the following somewhat more general question:

- For any $n \geq 1$, how can we decide which elements of G_n have a minimal reduction (in G_n)?

At this point we will begin to make use of the assumption we have made about the multiplicity matrices $\overline{A}_{n,n+1}$. Namely, that each has full rank. Under such circumstances we can guarantee that a graph $\Gamma \in G_n$ has a minimal reduction for all $n \geq 1$.

Theorem 3.5. *Let $\Gamma \in G_n$ for any $n \geq 1$ and suppose M_Γ is the multiplicity matrix which describes Γ. If $\mathrm{rank}(M_\Gamma) = n$ then Γ has a minimal reduction.*

Proof. We will proceed via induction on n. The induction basis is provided by the case $n = 1$, where it is easy to see that all graphs $\Gamma \in G_1$ have minimal reductions.

Next, suppose the result holds for n and let $\Gamma \in G_{n+1}$. We will assume $\mathrm{rank}(M_\Gamma) = n + 1$ and write $M_\Gamma = [a_{ij}]$, $1 \leq i \leq n + 2$, $1 \leq j \leq n+1$. Before proceeding further we remark that by permuting the rows and columns of M_Γ we are merely rearranging the vertices in

the graph. For example, a row permutation amounts to rearranging the vertices at level 2 and a column permutation amounts to rearranging the vertices at level 1. Of course, if it is possible to obtain a minimal reduction of this permuted form of the original graph, then by reversing the permutations, we will also have a minimal reduction of the original graph. We will therefore work with various matrices obtained from M_Γ through row and column permutations in order to obtain our result.

We consider two cases:

(a) There exists $j_0 \in \{1, \ldots, n+1\}$ such that the submatrix $[a_{ij}]$, $1 \le i \le n+2$, $1 \le j \le n+1$, $j \ne j_0$, has only nonzero rows.
(b) For every $j_0 \in \{1, \ldots, n+1\}$, the submatrix $[a_{ij}]$, $1 \le i \le n+2$, $1 \le j \le n+1$, $j \ne j_0$, has a zero row.

Consider case (a). Permute the columns of M_Γ so that the submatrix that results from omitting the last column of the permuted matrix has only nonzero rows. For notational convenience, we will continue to write M_Γ and $[a_{ij}]$ for these permuted forms of the original multiplicity matrix. Now, permute the rows so that the top $n+1$ rows are linearly independent, which we can do since $\operatorname{rank}(M_\Gamma) = n+1$. Thus, the submatrix $[a_{ij}]$, $1 \le i, j \le n+1$, is nonsingular.

Because $[a_{ij}]$, $1 \le i, j \le n+1$, is nonsingular, the matrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, has rank n. Thus, there exists i, which, without a loss of generality, we may suppose equals $n+1$, such that the submatrix $[a_{ij}]$, $1 \le i, j \le n$, is nonsingular.

At this point we make an assumption that is justified later (Lemma 3.6). Assume $a_{n+1,n+1} \ne 0$. Permute the last two rows of M_Γ to finally arrive at a matrix with the following characteristics:

(i) Omitting the $(n+1)$-st column leaves all nonzero rows;
(ii) The first n rows of the submatrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, are linearly independent; and
(iii) The entry $a_{n+2,n+1}$ is nonzero.

Since (i) and (ii) hold, the submatrix $[a_{ij}]$, $1 \le i \le n+1$, $1 \le j \le n$, is the multiplicity matrix for a graph in G_n with rank n. Thus, by the induction hypothesis, there exists a minimal reduction of this graph (which is just a subgraph of Γ). Since (iii) holds, there is at least one edge connecting the last vertex at level 1 with the last vertex at level 2. By deleting all other edges which connect these last two vertices to any others, we obtain a minimal reduction of Γ.

Next, consider case (b). The assumption implies that at least $n+1$ rows have exactly one nonzero entry and for different rows, the columns in which these entries appear are different. We may assume without a loss of generality that these rows are the first $n+1$ rows. So, the

first $n+1$ vertices at level 2 are connected to exactly one vertex at level 1, and different level 2 vertices are connected to different level 1 vertices. Finally, since the last row is not zero, the last vertex at level 2 is connected to at least one vertex at level 1. Thus, a minimal reduction in G_{n+1} is possible.

□

We now justify an assumption made in the proof of the previous theorem.

Lemma 3.6. *Given an invertible matrix $B = [b_{ij}] \in M_n(\mathbb{C})$, there exists k, $1 \leq k \leq n$, such that the submatrix $[b_{ij}]$, $1 \leq i, j \leq n$, $i \neq k$, $j \neq n$, is nonsingular and $b_{k,n} \neq 0$.*

Proof. First, if there exists $1 \leq k \leq n$ such that $b_{k,1} = \cdots = b_{k,n-1} = 0$, then it must be that $b_{k,n} \neq 0$. Furthermore, since the dimension of the set $\operatorname{span}\{[b_{i,1}, \ldots, b_{i,n-1}] : 1 \leq i \leq n\}$ is $n-1$, the desired result is achieved. To complete the proof we consider the case where $[b_{i,1}, \ldots, b_{i,n-1}] \neq 0$, for every $1 \leq i \leq n$.

Assume, without a loss of generality, that the first i_0 rows of the matrix B end in 0 (i.e., $b_{i,n} = 0$, for all $1 \leq i \leq i_0$) and the remaining rows end in a nonzero number (i.e., $b_{i,n} \neq 0$, for all $i_0 + 1 \leq i \leq n$). Of course, $i_0 < n$ since B is nonsingular. If we assume that there exist scalars $\alpha_1, \ldots, \alpha_{i_0}$ such that

$$\alpha_1[b_{1,1}, \ldots, b_{1,n-1}] + \cdots + \alpha_{i_0}[b_{i_0,1}, \ldots, b_{i_0,n-1}] = 0$$

then

$$\alpha_1[b_{1,1}, \ldots, b_{1,n-1}, b_{1,n}] + \cdots + \alpha_{i_0}[b_{i_0,1}, \ldots, b_{i_0,n-1}, b_{i_0,n}] = 0$$

as well since $b_{1,n} = \cdots = b_{i_0,n} = 0$. Because these later vectors are linearly independent, it must be that $\alpha_1 = \cdots = \alpha_{i_0} = 0$, and therefore, the set

$$\{[b_{i,1}, \ldots, b_{i,n-1}] : 1 \leq i \leq i_0\}$$

is linearly independent.

We know the set $\{[b_{i,1}, \ldots, b_{i,n-1}] : 1 \leq i \leq n\}$ is linearly dependent, and so there exist coefficients $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i[b_{i,1}, \ldots, b_{i,n-1}] = 0.$$

Furthermore, since $\{[b_{i,1}, \ldots, b_{i,n-1}] : 1 \leq i \leq i_0\}$ is a linearly independent set, it must be that at least one coefficient α_{i_1}, such that $i_1 > i_0$,

is nonzero. But then,

$$\begin{aligned}\mathbb{C}^{n-1} &= \operatorname{span}\{[b_{i,1},\ldots,b_{i,n-1}] : 1 \le i \le n\} \\ &= \operatorname{span}\{[b_{i,1},\ldots,b_{i,n-1}] : 1 \le i \le n, i \ne i_1\},\end{aligned}$$

which implies $\{[b_{i,1},\ldots,b_{i,n-1}] : 1 \le i \le n, i \ne i_1\}$ is linearly independent. Hence, the submatrix $[b_{ij}]$, $1 \le i \le n$, $i \ne i_1$, $1 \le j \le n-1$, is nonsingular and $b_{i_1,n} \ne 0$.

□

This proves that for those AF algebras which are of type (2), a minimal Bratteli diagram corresponding to the set X_{min} exists under the assumption that each multiplicity matrix has full rank. For algebras of type (1), we have an analogous result.

Theorem 3.7. *Suppose* Γ *is a graph consisting of* n *vertices at both levels* 1 *and* 2 *and* (II) *and* (III) *of Properties* 3.1 *are satisfied. If the multiplicity matrix* M_Γ *which describes the edges has rank* n*, then* Γ *has a minimal reduction* (*a reduction with* n *edges*) *which satisfies* (II) *and* (III) *of Properties* 3.1*.*

Proof. The proof is very similar to that of Theorem 3.5.

□

The results obtained here will be used in the subsequent section to prove a result about the K_0 groups of a large class of AF algebras. Despite that motive for their inclusion, they are interesting in their own right. After all, being able to reduce a Bratteli diagram in the way described here means that a sub-Bratteli diagram exists which is minimal in some sense. Specifically, deleting any more edges will result in a subgraph which is no longer itself a Bratteli diagram. Of course, all Bratteli diagrams are reducible to sub-Bratteli diagrams (possibly in a trivial way). However, the reductions here to the level of X_{min} are as far as one can go. Deleting any more edges, without deleting any vertices, will result in a subgraph which is no longer a Bratteli diagram.

4. Dimension Groups and Minimal Bratteli Diagrams

As is well known, if X is a 0-dimensional (basis consisting of clopen sets) compact metric space, then there exists a sequence $\{E_n\}_{n=0}^{\infty}$ of successively finer partitions of X which generate the topology and consist of clopen sets. Therefore, $\{C(E_n)\}_{n=0}^{\infty}$, where $C(E_n)$ consists of those functions constant on elements of the partition E_n, is an increasing sequence of finite-dimensional C*-algebras (isomorphic to $\mathbb{C}^{|E_n|}$). Since $C(X) = \overline{\bigcup_{n\ge 0} C(E_n)}$, it follows that $C(X)$ is AF.

For any $n \geq 0$, the dimension group $K_0(C(E_n))$ of $C(E_n)$ is easily seen to be isomorphic to $(C(E_n, \mathbb{Z}), C(E_n, \mathbb{Z}^+), \chi_X)$, where $C(E_n, \mathbb{Z})$ are the continuous functions from X to $\mathbb{Z}$ constant on the elements of E_n. As such, one can conclude that

$$K_0(C(X)) = \varinjlim K_0(C(E_n)) \cong \left(C(X, \mathbb{Z}), C(X, \mathbb{Z}^+), \chi_X\right).$$

One aspect of this well known example which we want to draw attention to is that $X_{min} = X$. Therefore, in a natural way, the dimension group of $C(X)$ can be realized as a group of continuous functions on X_{min}. In the context of AF groupoids, [14] presents additional examples for which this also holds true. We intend in this section to generalize the results of [14] in order to demonstrate that the dimension groups of those AF algebras in a certain class can be realized in this same way, as groups of continuous functions on X_{min}.

Here, as in the previous section, we begin with an AF algebra $\mathfrak{A}$ such that the sequence $\{m_n\}_{n=0}^{\infty}$ is strictly monotonically increasing and each multiplicity matrix $\overline{A}_{n,n+1}$ has full rank. Of course, by one of our results, we may assume without a loss of generality that $m_n = n+1$, for all $n \geq 0$.

By Theorem 3.5 we can reduce the Bratteli diagram representing $\mathfrak{A}$ to a sub-Bratteli diagram whose set of all infinite paths corresponds to X_{min}, as in the discussion of Section 2. Put in somewhat more concrete terms, Theorem 3.5 tells us that the standard embeddings can be chosen so that for each $n \geq 1$, there exist integers $1 \leq r'_n < r_n \leq n+1$ and $1 \leq a_n \leq n$ with

$$\widehat{e}_1(r'_n, n) \cup \widehat{e}_1(r_n, n) \subset \widehat{e}_1(a_n, n-1),$$

and that there exists a bijection

$$\sigma : \{i : 1 \leq i \leq n+1, i \neq r'_n, r_n\} \to \{j : 1 \leq j \leq n, j \neq a_n\}$$

such that $\widehat{e}_1(i, n) \subset \widehat{e}_1(\sigma(i), n-1)$.

To achieve our result, we will now define, for all $n \geq 1$, a linear map $R_n : \mathbb{C}^{n+1} \to C(X_{min}, \mathbb{C})$ by

$$R_n(\alpha_1, \ldots, \alpha_{n+1}) = \sum_{l=0}^{n} \alpha_{l+1} \chi_{B(r_l, l)},$$

where $r_0 = 1$ and, as a notational convenience, we write $B(i, n)$ to denote the set $\widehat{e}_1(i, n) \cap X_{min}$. Note that we are now assuming that a specific choice for X_{min} has been made. Thus, the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{r'_n\}_{n=1}^{\infty}$ are fixed. The following technical result about the maps R_n will be useful to us in what follows.

Lemma 4.1. *For all $n \geq 1$, if $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{C}^{n+1}$ then there exists $(\beta_1, \ldots, \beta_{n+1})^T \in \mathbb{C}^{n+1}$ such that*

$$R_n(\beta_1, \ldots, \beta_{n+1}) = \sum_{l=1}^{n+1} \alpha_l \chi_{B(l,n)}.$$

Proof. For $n = 1$, set $f = \alpha_1 \chi_{B(1,1)} + \alpha_2 \chi_{B(2,1)}$. Let $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 - \alpha_1$. Then,

$$\begin{aligned} R_1(\beta_1, \beta_2) &= \beta_1 \chi_{B(1,0)} + \beta_2 \chi_{B(2,1)} \\ &= \alpha_1 \chi_{B(1,0)} + (\alpha_2 - \alpha_1) \chi_{B(2,1)}. \end{aligned}$$

We note that $B(1,1) \cup B(2,1) = B(1,0)$, and so,

$$R_1(\beta_1, \beta_2) = \alpha_1 \chi_{B(1,1)} + \alpha_1 \chi_{B(2,1)} + (\alpha_2 - \alpha_1) \chi_{B(2,1)} = f.$$

Now, assume that for every $f = \sum_{l=1}^{n+1} \alpha_l \chi_{B(l,n)}$ there exists a vector $(\beta_1, \ldots, \beta_{n+1})^T$ such that $R_n(\beta_1, \ldots, \beta_{n+1}) = f$, where the β_i are linear combinations of the α_j. Let $g = \sum_{l=1}^{n+2} \gamma_l \chi_{B(l,n+1)}$. By our earlier comments, there exist $1 \leq r'_{n+1} < r_{n+1} \leq n+2$ and $1 \leq a_{n+1} \leq n+1$ such that $B(r'_{n+1}, n+1) \cup B(r_{n+1}, n+1) = B(a_{n+1}, n)$. Furthermore, there exists a bijection σ such that $B(i, n+1) = B(\sigma(i), n)$. Then,

$$\begin{aligned} g &= \sum_{l=1}^{n+2} \gamma_l \chi_{B(l,n+1)} = \sum_{l=1, l \neq r_{n+1}, r'_{n+1}}^{n+2} \gamma_l \chi_{B(\sigma(l),n)} \\ &+ \gamma_{r_{n+1}} \chi_{B(r_{n+1},n+1)} + \gamma_{r'_{n+1}} \chi_{B(r'_{n+1},n+1)} \\ &= \sum_{l=1, l \neq r_{n+1}}^{n+2} \gamma_l \chi_{B(\sigma(l),n)} + (\gamma_{r_{n+1}} - \gamma_{r'_{n+1}}) \chi_{B(r_{n+1},n+1)} \end{aligned}$$

where we define $\sigma(r'_{n+1}) = a_{n+1}$. By the induction hypothesis, there exists $(\beta_1, \ldots, \beta_{n+1})$ such that

$$\sum_{l=1, l \neq r_{n+1}}^{n+2} \gamma_l \chi_{B(\sigma(l),n)} = R_n(\beta_1, \ldots, \beta_{n+1}).$$

Thus,

$$\begin{aligned} g &= R_n(\beta_1, \ldots, \beta_{n+1}) + (\gamma_{r_{n+1}} - \gamma_{r'_{n+1}}) \chi_{B(r_{n+1},n+1)} \\ &= \sum_{l=0}^{n} \beta_{l+1} \chi_{B(r_l,l)} + (\gamma_{r_{n+1}} - \gamma_{r'_{n+1}}) \chi_{B(r_{n+1},n+1)} \\ &= \sum_{l=0}^{n+1} \beta_{l+1} \chi_{B(r_l,l)} = R_{n+1}(\beta_1, \ldots, \beta_{n+2}), \end{aligned}$$

where $\beta_{n+2} = \gamma_{r_{n+1}} - \gamma_{r'_{n+1}}$. Hence, by induction, the result holds for all $n \geq 1$.

□

We note that R_n injects for all $n \geq 1$. This follows from the fact that by construction, the vector $(\beta_1, \ldots, \beta_{n+1})^T$ which accomplishes $R_n(\beta_1, \ldots, \beta_{n+1}) = 0$ has components which are linear combinations of zeros. Therefore, $\beta_1 = \cdots = \beta_{n+1} = 0$.

Now since, by assumption, $\overline{A}_{n,n+1}$ has full rank, for all $n \geq 0$, it is possible, by adding appropriate columns $[a_{1,n+2}, \ldots, a_{n+2,n+2}]^T \in \mathbb{N}^{n+2}$ to create a sequence $\{A_{n,n+1}\}_{n=0}^{\infty}$ of nonsingular matrices where

$$A_{n,n+1} = \left[\begin{array}{cc} & a_{1,n+2} \\ \overline{A}_{n,n+1} & \vdots \\ & a_{n+2,n+2} \end{array} \right] \in M_{n+2}(\mathbb{N}).$$

Remark 4. At this point, any choice for $[a_{1,n+2}, \ldots, a_{n+2,n+2}]^T$ which makes $A_{n,n+1}$ nonsingular is appropriate. In fact, there is no *a priori* reason that the $a_{i,n+2}$ can not be elements of $\mathbb{C}$. However, as we will see in a moment, in certain instances a somewhat more restrictive choice will be desirable.

To compute the dimension group $K_0(\mathfrak{A})$, we utilize the fact that $K_0(\mathfrak{A}) = \varinjlim K_0(\mathfrak{A}_n)$. However, to be more explicit about the nature of $\varinjlim K_0(\mathfrak{A}_n)$, we first define, for all $n \geq 1$,

$$A_n = \left[A_{0,1}^{-1} \oplus I_{n-1}\right] \cdots \left[A_{n-2,n-1}^{-1} \oplus I_1\right] A_{n-1,n}^{-1}.$$

Then, let $\Phi_n : \mathbb{Z}^{n+1} \to C(X_{min}, G)$ be given by $\Phi_n = R_n \circ A_n$, for all $n \geq 1$, where G is an abelian group that will now be described. Of course, due to the definition of R_n, the nature of G depends on the range of A_n when its domain is taken as $\mathbb{Z}^{n+1}$.

Letting $n \geq 1$ be given, it is clear that

$$A_n^{-1} = A_{n-1,n} \left[A_{n-2,n-1} \oplus I_1\right] \cdots \left[A_{1,2} \oplus I_{n-2}\right] \left[A_{0,1} \oplus I_{n-1}\right]$$

and, by the multiplicativity of the determinant, that

$$\left|A_n^{-1}\right| = \left|A_{n-1,n}\right| \cdot \left|A_{n-2,n-1}\right| \cdot \cdots \cdot \left|A_{1,2}\right| \cdot \left|A_{0,1}\right|.$$

As, for example, in [12, pages 20–21], we can then write

$$A_n = \frac{1}{\left|A_n^{-1}\right|} \operatorname{adj}\left(A_n^{-1}\right),$$

where $\operatorname{adj}(A_n^{-1})$ is the *adjugate* or *classical adjoint* of the matrix A_n^{-1}. In defining each matrix $A_{n,n+1}$, if we choose columns of integers, then each matrix A_n^{-1} will be a product of matrices with integer entries,

and therefore itself must have integer entries. But, by the definition of $\mathrm{adj}(A_n^{-1})$, it will then follow that $\mathrm{adj}(A_n^{-1})$ has integer entries as well.

We will now define the set G_n by

$$G_n = \left\{ \frac{a}{|A_n^{-1}|} : a \in \mathbb{Z} \right\}.$$

Then, G_n is embedded in G_{n+1} by inclusion, and we write

$$G = \varinjlim G_n = \bigcup_{n \geq 1} G_n \subset \mathbb{Q}.$$

It is then clear that for $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$,

$$A_n(\alpha_1, \ldots, \alpha_{n+1})^T \in G_{n+1}.$$

Giving G the discrete topology, we therefore see that Φ_n will be a map from $\mathbb{Z}^{n+1}$ to $C(X_{min}, G)$.

To show that $K_0(\mathfrak{A})$ is (at least) a subgroup of $C(X_{min}, G)$, we would like to show that the diagram

$$\begin{tikzcd}
K_0(\mathfrak{A}_n) \cong \mathbb{Z}^{n+1} \arrow[r, "\phi_{n*}"] \arrow[dr, "\Phi_n"'] & K_0(\mathfrak{A}_{n+1}) \cong \mathbb{Z}^{n+2} \arrow[d, "\Phi_{n+1}"] \\
& C(X_{min}, G)
\end{tikzcd}$$

commutes, where ϕ_{n*} is the homomorphism induced by the standard embedding $\phi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$.

Let $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$. Then, $\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T$ can be written as

$$\left(R_{n+1} \circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-1,n}^{-1} \oplus I_1] A_{n,n+1}^{-1}\right) \circ A_{n,n+1}(\alpha_1, \ldots, \alpha_{n+1}, 0)^T.$$

If we define $(\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1})^T = A_{n-1,n}^{-1}(\alpha_1, \ldots, \alpha_{n+1})^T$ and, in general, for $2 \leq i < n$,

$$(\beta_1^{n-i}, \ldots, \beta_{n-i+2}^{n-i})^T = A_{n-i,n-i+1}^{-1}(\beta_1^{n-i+1}, \ldots, \beta_{n-i+2}^{n-i+1})^T,$$

we see that

$$\begin{aligned}
&\Phi_{n+1} \circ \phi_{n*}(\alpha_1, \ldots, \alpha_{n+1})^T \\
&\quad = R_{n+1} \circ [A_{0,1}^{-1} \oplus I_n] \cdots [A_{n-2,n-1}^{-1} \oplus I_2](\beta_1^{n-1}, \ldots, \beta_{n+1}^{n-1}, 0)^T \\
&\quad = R_{n+1}(\beta_1^0, \beta_2^0, \beta_3^1, \ldots, \beta_n^{n-2}, \beta_{n+1}^{n-1}, 0) \\
&\quad = \beta_1^0 \chi_{B(1,0)} + \sum_{l=1}^{n} \beta_{l+1}^{l-1} \chi_{B(r_l,l)}.
\end{aligned}$$

However, the following calculation yields

$$\begin{aligned}\Phi_n(\alpha_1,\ldots,\alpha_{n+1})^T &= R_n(\beta_1^0,\beta_2^0,\beta_3^1,\ldots,\beta_n^{n-2},\beta_{n+1}^{n-1}) \\ &= \beta_1^0\chi_{B(1,0)} + \sum_{l=1}^{n}\beta_{l+1}^{l-1}\chi_{B(r_l,l)}.\end{aligned}$$

Thus, $\Phi_n = \Phi_{n+1}\circ\phi_{n*}$, and we conclude that the diagram commutes. By the universal property of the direct limit, it follows that there exists a homomorphism $\Psi : K_0(\mathfrak{A}) \to C(X_{min}, G)$. Since each of the maps Φ_n are injective, it follows that in fact, Ψ is an injection.

Remark 5. Considering $C(X_{min}, G)$ as an ordered group with positive cone $C(X_{min}, G^+)$, this homomorphism is not necessarily order preserving. That is, in general,

$$K_0^+(\mathfrak{A}) \ncong K_0(\mathfrak{A}) \cap C(X_{min}, G^+).$$

We will say more about this in a moment. It is true, however, that the order unit of $K_0(\mathfrak{A})$ in $C(X_{min}, G)$ is $\chi_{X_{min}}$.

The following theorem provides conditions under which more information about the structure of $K_0(\mathfrak{A})$ is available.

Theorem 4.2. *If, in constructing the matrices $A_{n,n+1}$, integer-valued columns can be chosen so that $|A_{n,n+1}| = 1$, for all $n \geq 0$, then $K_0(\mathfrak{A}) \cong C(X_{min}, \mathbb{Z})$ and the order unit is $\chi_{X_{min}}$. In general, however, $K_0^+(\mathfrak{A}) \ncong C(X_{min}, \mathbb{Z}^+)$.*

Proof. In this case each matrix A_n is invertible and has integer entries. Thus, $K_0(\mathfrak{A})$ is a subgroup of $C(X_{min}, \mathbb{Z})$. If we let $f \in C(X_{min}, \mathbb{Z})$ then by continuity and the compactness of X_{min}, there exists $n \geq 1$ and $(\alpha_1,\ldots,\alpha_{n+1})^T \in \mathbb{Z}^{n+1}$ such that

$$f = \sum_{l=1}^{n+1}\alpha_l\chi_{B(l,n)}.$$

By Lemma 4.1, there exists $(\beta_1,\ldots,\beta_{n+1})^T \in \mathbb{Z}^{n+1}$ such that $f = R_n(\beta_1,\ldots,\beta_{n+1})$. Thus,

$$A_n^{-1}(\beta_1,\ldots,\beta_{n+1})^T \in \mathbb{Z}^{n+1} \cong K_0(\mathfrak{A}_n),$$

and we have $f = \Phi_n(A_n^{-1}(\beta_1,\ldots,\beta_{n+1})^T)$.

□

This theorem implies that whenever two AF algebras satisfy the hypotheses of Theorem 4.2 and their standard embeddings can be chosen to yield the same X_{min}, then the only aspect of their dimension groups which distinguishes them is the positive cone. This, of course, makes

it clear why, in general, $K_0^+(\mathfrak{A})$ is not $C(X_{min}, \mathbb{Z}^+)$. At the beginning of this section we commented on how if X is a 0-dimensional compact metric space then the AF algebra $C(X)$ has dimension group $(C(X, \mathbb{Z}), C(X, \mathbb{Z}^+), \chi_X)$. Therefore, given $\mathfrak{A}$ satisfying the hypotheses of Theorem 4.2, for any X_{min} associated with $\mathfrak{A}$,

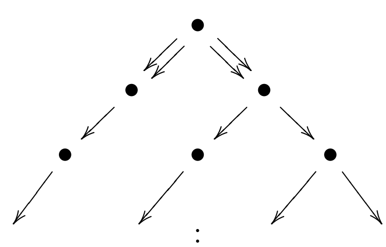
$$K_0(C(X_{min})) \cong (C(X_{min}, \mathbb{Z}), C(X_{min}, \mathbb{Z}^+), \chi_{X_{min}}).$$

So, by Elliott's Theorem [9] and Theorem 4.2, unless $C(X_{min}) \cong \mathfrak{A}$, it must be that $K_0^+(\mathfrak{A}) \subset C(X_{min}, \mathbb{Z})$ is distinct from $C(X_{min}, \mathbb{Z}^+)$.

One also sees that the flexibility that may exist in choosing X_{min} is of no help here. That is, any choice for X_{min} will result in $K_0^+(\mathfrak{A})$ being in general different from $C(X_{min}, \mathbb{Z}^+)$. Despite this, [14] shows that useful characterizations of $K_0^+(\mathfrak{A})$ as a subset of $C(X_{min}, \mathbb{Z})$ do exist in specific cases. Furthermore, circumstances under which the topological structure of X_{min} is useful in distinguishing between AF algebras will be outlined in Theorem 4.4 below.

Earlier, in showing that $K_0(\mathfrak{A})$ is a subgroup of $C(X_{min}, G)$, we alluded to the fact that it may be strictly contained in $C(X_{min}, G)$. The following example confirms this.

Example 4.3. Consider the AF algebra $\mathfrak{A}$ with Bratteli diagram



Here, $\overline{A}_{0,1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and, for $n \geq 1$,

$$\overline{A}_{n,n+1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & 1 \end{bmatrix},$$

where all unspecified entries are zero. Suppose $A_{0,1} = \begin{bmatrix} 2 & a_{1,2} \\ 2 & a_{2,2} \end{bmatrix}$ and, for $n \geq 1$,

$$A_{n,n+1} = \begin{bmatrix} 1 & & & a_{1,n+1} \\ & \ddots & & \vdots \\ & & 1 & a_{n,n+1} \\ & & 1 & a_{n+1,n+1} \end{bmatrix},$$

where the columns are chosen in order to make $A_{n,n+1}$ invertible for all $n \geq 0$. Note that this means $a_{2,2} - a_{1,2} \neq 0$. We will show that for all $n \geq 1$, the vector $\left(0, \frac{1}{2(a_{2,2} - a_{1,2})}, 0, \ldots, 0\right)^T \in G_{n+1}$ is not in the range of A_n. Of course, since A_n is nonsingular, this is equivalent to showing that

$$A_n^{-1}\left(0, \frac{1}{2(a_{2,2} - a_{1,2})}, 0, \ldots, 0\right)^T \notin \mathbb{Z}^{n+1},$$

for all $n \geq 1$.

It can be easily checked that $A_n^{-1}\left(0, \frac{1}{2(a_{2,2} - a_{1,2})}, 0, \ldots, 0\right)^T$ equals

$$\frac{1}{2(a_{2,2} - a_{1,2})}(a_{1,2}, a_{2,2}, a_{2,2}, \ldots, a_{2,2})^T.$$

Now, if $\frac{a_{1,2}}{2(a_{2,2} - a_{1,2})}, \frac{a_{2,2}}{2(a_{2,2} - a_{1,2})} \in \mathbb{Z}$ then

$$\frac{a_{2,2}}{2(a_{2,2} - a_{1,2})} - \frac{a_{1,2}}{2(a_{2,2} - a_{1,2})} = \frac{1}{2}$$

should also be an element of $\mathbb{Z}$. This contradiction implies that there exists a function in $C(X_{min}, G)$ not in the range of Φ_n for any n. Hence, $K_0(\mathfrak{A})$ is a strict subgroup of $C(X_{min}, G)$.

The next important theorem provides conditions under which the topological structure of X_{min} can be used to help discriminate between AF algebras.

Theorem 4.4. *Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are AF algebras such that $K_0(\mathfrak{A}_i) \subset C(X^i_{min}, G_i)$ and $K_0^+(\mathfrak{A}_i) \subset C(X^i_{min}, G_i^+)$, for $i = 1, 2$, where both G_1 and G_2 are subgroups of $\mathbb{Q}$ as in the discussion preceding Theorem 4.2. Assume that for all $U \subset X^i_{min}$ clopen, $\chi_U \in K_0(\mathfrak{A}_i)$. If $\psi : K_0(\mathfrak{A}_1) \to K_0(\mathfrak{A}_2)$ is an order preserving isomorphism taking order unit to order unit, then X^1_{min} and X^2_{min} are homeomorphic.*

Proof. To begin, let $x \in X^1_{min}$ and take $\{U_n\}_{n=1}^\infty$ to be a decreasing sequence of clopen subsets in X^1_{min} with $\{x\} = \bigcap_{n=1}^\infty U_n$. Then,

$$\chi_{X^2_{min}} = \psi(\chi_{X^1_{min}}) = \psi(\chi_{U_n}) + \psi(\chi_{U_n^c}),$$

for all $n \geq 1$, where U_n^c is the complement of U_n in X^1_{min}. If we suppose the supports of the functions $\psi(\chi_{U_n})$ and $\psi(\chi_{U_n^c})$ are not disjoint, then there exists a nonempty clopen set $V \subset X^2_{min}$ such that $\alpha\chi_V \leq \psi(\chi_{U_n}), \psi(\chi_{U_n^c})$, where $\alpha \in G_2^+$, $\alpha \neq 0$. Because ψ^{-1} is order preserving, it follows that

$$\psi^{-1}(\alpha\chi_V) \leq \chi_{U_n}, \chi_{U_n^c},$$

implying $\psi^{-1}(\alpha\chi_V) = 0$ since $U_n \cap U_n^c = \emptyset$. Thus, $\alpha\chi_V = 0$, a contradiction, and we conclude that $\psi(\chi_{U_n})$ and $\psi(\chi_{U_n^c})$ have disjoint supports. Therefore, by continuity there exists a nonempty clopen subset $V_n \subset X^2_{min}$ such that $\psi(\chi_{U_n}) = \chi_{V_n}$, for all $n \geq 1$. Because $\{\chi_{U_n}\}_{n=1}^\infty$ is a decreasing sequence in $C(X^1_{min}, G_1)$, $\{\chi_{V_n}\}_{n=1}^\infty$ decreases in $C(X^2_{min}, G_2)$, and consequently, $\{V_n\}_{n=1}^\infty$ is a decreasing sequence of clopen sets in X^2_{min}.

Let $y \in \bigcap_{n=1}^\infty V_n$ and let $\{W_n\}_{n=1}^\infty$ be a decreasing sequence of clopen sets in X^2_{min} such that $\{y\} = \bigcap_{n=1}^\infty W_n$ and $W_n \subset V_n$, for all $n \geq 1$. Then, $\psi^{-1}(\chi_{W_n}) \leq \chi_{U_n}$, and letting $\psi^{-1}(\chi_{W_n}) = \chi_{Z_n}$, Z_n clopen, we have $Z_n \subset U_n$, for all $n \geq 1$. Furthermore, $x \in Z_n$, for all n since otherwise Z_n and U_n would eventually be disjoint. Hence, $\{x\} = \bigcap_{n=1}^\infty Z_n$.

Now, if $y_1 \neq y_2$, $y_1, y_2 \in \bigcap_{n=1}^\infty V_n$, then there would exist sequences $\{W_n^1\}$, $\{W_n^2\}$ of clopen sets, such that $y_1 \in \bigcap_{n=1}^\infty W_n^1$, $y_2 \in \bigcap_{n=1}^\infty W_n^2$, and $W_n^1 \cap W_n^2 = \emptyset$, for all $n \geq 1$. But then $\psi^{-1}(\chi_{W_n^1})$ and $\psi^{-1}(\chi_{W_n^2})$ would have disjoint supports, and so the corresponding sets Z_n^1, Z_n^2, would be disjoint. This contradicts $\{x\} = \bigcap_{n=1}^\infty Z_n^i$, $i = 1, 2$. Hence, $\{y\} = \bigcap_{n=1}^\infty V_n$.

Define $f : X^1_{min} \to X^2_{min}$ by letting $f(x) = y$, for all $x \in X^1_{min}$. Clearly this map is well defined since any two sequences $\{U_n\}$, $\{U_n'\}$ which decrease down to $\{x\}$ can be intertwined, resulting in the intertwining of the corresponding sequences $\{V_n\}$ and $\{V_n'\}$. It is also clear that f is a bijection. Finally, for U clopen in X^1_{min}, one sees that if $\psi(\chi_U) = \chi_V$ then $f(U) = V$. Therefore, f is an open map, and similarly so is its inverse. Hence, X^1_{min} is homeomorphic to X^2_{min}.

□

Remark 6. This theorem generalizes the well-known fact that if $C(X_1)$ and $C(X_2)$ are isomorphic AF algebras, then their spectra X_1 and X_2 are homeomorphic.

To this point, we have only been concerned with those AF algebras $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$ such that $m_n = n + 1$, for all $n \geq 0$. The case where $m_n = L \in \mathbb{Z}^+$, for all $n \geq 0$ and L fixed, is simpler and more transparent. For completeness, however, we mention briefly the key points.

Here, under the assumption that $\overline{A}_{n,n+1} \in M_L(\mathbb{N})$ is invertible, any minimal Bratteli diagram corresponding to the set X_{min} will be homeomorphic to a finite set $\{x_1, \ldots, x_L\}$ with the discrete topology by Theorem 3.7. Then, define the maps

$$\overline{\Phi}_n = \overline{R} \circ \left[\overline{A}_{0,1}^{-1} \cdots \overline{A}_{n-1,n}^{-1}\right] : K_0(\mathfrak{A}_n) \to C(X_{min}, G)$$

where $\overline{R}(\alpha_1, \ldots, \alpha_L)^T = \sum_{i=1}^{L} \alpha_i \chi_{\{x_i\}}$,

$$G_n = \left\{ \frac{a}{\left|\overline{A}_{0,1}\right| \cdots \left|\overline{A}_{n-1,n}\right|} : a \in \mathbb{Z} \right\},$$

and $G = \varinjlim G_n$. It follows that the diagram

$$\begin{tikzcd} K_0(\mathfrak{A}_n) \cong \mathbb{Z}^L \arrow[r, "\phi_{n*}"] \arrow[dr, "\overline{\Phi}_n"'] & K_0(\mathfrak{A}_{n+1}) \cong \mathbb{Z}^L \arrow[d, "\overline{\Phi}_{n+1}"] \\ & C(X_{min}, G) \end{tikzcd}$$

commutes, and therefore, $K_0(\mathfrak{A})$ is a subgroup of $C(X_{min}, G)$.

As with the earlier discussion, in general, $K_0(\mathfrak{A})$ is a strict subgroup of $C(X_{min}, G)$, the order unit is $\chi_{X_{min}}$, and the positive cone is not necessarily $C(X_{min}, G^+)$ (or even a subset of it). We also note here that Theorems 4.2 and 4.4 remain true in this context.

5. Unique Minimal Bratteli Diagrams

Considering those AF algebras $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$ such that $\{m_n\}_{n=0}^{\infty}$ is strictly monotonically increasing, we assume now that $\mathfrak{A}$ has a Bratteli diagram with a unique subgraph corresponding to the set X_{min}. That is to say, the Bratteli diagram has a unique reduction to a minimal diagram. As above, for convenience we dilate the sequence and assume without a loss of generality that $m_n = n + 1$ for all $n \geq 0$. It is clear that this does not alter the fact that the diagram corresponding to X_{min} is unique. One can then show, up to unitary equivalence, that the connecting homomorphisms correspond to multiplicity matrices of

the form

$$\overline{A}_{n,n+1} = \begin{bmatrix} a^n_{1,1} & & & & & \\ & \ddots & & & & \\ & & a^n_{j(n),j(n)} & & & \\ & & a^n_{j(n)+1,j(n)} & & & \\ & & & a^n_{j(n)+2,j(n)+1} & & \\ & & & & \ddots & \\ & & & & & a^n_{n+2,n+1} \end{bmatrix},$$

for some $1 \leq j(n) \leq n+1$ and where all unspecified entries are zero. Therefore, by choosing nonzero integers b_n $(= a_{j(n)+1,n+2})$, $n \geq 0$, we can define, for all $n \geq 0$,

$$A_{n,n+1} = \begin{bmatrix} a^n_{1,1} & & & & & & \\ & \ddots & & & & & \\ & & a^n_{j(n),j(n)} & & & & \\ & & a^n_{j(n)+1,j(n)} & & & & b_n \\ & & & a^n_{j(n)+2,j(n)+1} & & & \\ & & & & \ddots & & \\ & & & & & a^n_{n+2,n+1} & \end{bmatrix},$$

in which case $A^{-1}_{n,n+1}$ is the matrix

$$\begin{bmatrix} {a^n_{1,1}}^{-1} & & & & & & \\ & \ddots & & & & & \\ & & {a^n_{j(n),j(n)}}^{-1} & 0 & & & \\ & & & 0 & {a^n_{j(n)+2,j(n)+1}}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & {a^n_{n+2,n+1}}^{-1} \\ & & -\frac{a^n_{j(n)+1,j(n)}}{b_n a^n_{j(n),j(n)}} & \frac{1}{b_n} & & & 0 \end{bmatrix}.$$

Now, letting $(\alpha_1, \alpha_2)^T \in \mathbb{Z}^2$, we see that

$$\begin{aligned} \Phi_1(\alpha_1,\alpha_2)^T &= R_1\left(\frac{\alpha_1}{a^0_{1,1}}, \frac{\alpha_2}{b_0} - \frac{\alpha_1 a^0_{2,1}}{a^0_{1,1} b_0}\right)^T \\ &= \frac{\alpha_1}{a^0_{1,1}}\chi_{B(1,0)} + \left(\frac{\alpha_2}{b_0} - \frac{\alpha_1 a^0_{2,1}}{a^0_{1,1} b_0}\right)\chi_{B(2,1)} \\ &= \frac{\alpha_1}{a^0_{1,1}}\chi_{B(1,1)} + \left(\frac{\alpha_1}{a^0_{1,1}} + \frac{\alpha_2}{b_0} - \frac{\alpha_1 a^0_{2,1}}{a^0_{1,1} b_0}\right)\chi_{B(2,1)}, \end{aligned}$$

since $B(1,0) = B(1,1) \cup B(2,1)$. Thus,

$$\Phi_1(\alpha_1, \alpha_2)^T = \frac{\alpha_1}{a^0_{1,1}} \chi_{B(1,1)} + \frac{\alpha_2}{b_0} \chi_{B(2,1)}$$

if we choose $b_0 = a^0_{2,1}$. We would like to show that there exists a sequence $\{b_n\}_{n=0}^{\infty}$ of positive integers such that for every $n \geq 1$,

$$\Phi_n(\alpha_1, \dots, \alpha_{n+1})^T = \sum_{i=1}^{n+1} \frac{\alpha_i}{k(i,n)} \chi_{B(i,n)},$$

where $k(i,n) \in \mathbb{Z}^+$, for all $1 \leq i \leq n+1$. Our above calculations prove that it is possible to choose b_0 accordingly.

Assume that $b_0, \dots, b_{k-1} \in \mathbb{Z}^+$ have been chosen in order to guarantee Φ_k is of the appropriate form for $1 \leq k \leq n$. Let $(\alpha_1, \dots, \alpha_{n+2})^T \in \mathbb{Z}^{n+2}$. Then,

$$\Phi_{n+1}(\alpha_1, \dots, \alpha_{n+2})^T = R_{n+1} \begin{bmatrix} A_n & 0 \\ 0 & 1 \end{bmatrix} A^{-1}_{n,n+1}(\alpha_1, \dots, \alpha_{n+2})^T.$$

By then first computing $A^{-1}_{n,n+1}(\alpha_1, \dots, \alpha_{n+2})^T$, one is able to show that $\Phi_{n+1}(\alpha_1, \dots, \alpha_{n+2})^T$ is equal to

$$\begin{aligned} &\Phi_n \left(\frac{\alpha_1}{a^n_{1,1}}, \dots, \frac{\alpha_{j(n)}}{a^n_{j(n),j(n)}}, \frac{\alpha_{j(n)+2}}{a^n_{j(n)+2,j(n)+1}}, \dots, \frac{\alpha_{n+2}}{a^n_{n+2,n+1}} \right) \\ + &\left(\frac{\alpha_{j(n)+1}}{b_n} - \frac{\alpha_{j(n)} a^n_{j(n)+1,j(n)}}{b_n a^n_{j(n),j(n)}} \right) \chi_{B(r_{n+1},n+1)}. \end{aligned}$$

Hence, our induction hypothesis allows us to write $\Phi_{n+1}(\alpha_1, \dots, \alpha_{n+2})^T$ as

$$\begin{aligned} &\sum_{i=1}^{j(n)} \frac{\alpha_i}{a^n_{i,i} k(i,n)} \chi_{B(i,n)} + \sum_{i=j(n)+1}^{n+1} \frac{\alpha_{i+1}}{a^n_{i+1,i} k(i,n)} \chi_{B(i,n)} \\ + &\left(\frac{\alpha_{j(n)+1}}{b_n} - \frac{\alpha_{j(n)} a^n_{j(n)+1,j(n)}}{b_n a^n_{j(n),j(n)}} \right) \chi_{B(r_{n+1},n+1)}. \end{aligned}$$

Now, we note that for $1 \leq i \leq j(n) - 1$, $B(i,n) = B(i,n+1)$, and for $j(n)+1 \leq i \leq n+1$, $B(i,n) = B(i+1,n+1)$. Furthermore, $B(j(n),n) = B(j(n),n+1) \cup B(j(n)+1,n+1)$ and $r_{n+1} = j(n)+1$.

Thus, $\Phi_{n+1}(\alpha_1, \ldots, \alpha_{n+2})^T$ becomes

$$\begin{aligned}
& \sum_{i=1}^{j(n)-1} \frac{\alpha_i}{a_{i,i}^n k(i,n)} \chi_{B(i,n+1)} + \sum_{i=j(n)+1}^{n+1} \frac{\alpha_{i+1}}{a_{i+1,i}^n k(i,n)} \chi_{B(i+1,n+1)} \\
+ & \frac{\alpha_{j(n)}}{a_{j(n),j(n)}^n k(j(n),n)} \left(\chi_{B(j(n),n+1)} + \chi_{B(j(n)+1,n+1)} \right) \\
+ & \left(\frac{\alpha_{j(n)+1}}{b_n} - \frac{\alpha_{j(n)} a_{j(n)+1,j(n)}^n}{b_n a_{j(n),j(n)}^n} \right) \chi_{B(j(n)+1,n+1)} \\
= & \sum_{i=1}^{j(n)} \frac{\alpha_i}{a_{i,i}^n k(i,n)} \chi_{B(i,n+1)} + \sum_{i=j(n)+1}^{n+1} \frac{\alpha_{i+1}}{a_{i+1,i}^n k(i,n)} \chi_{B(i+1,n+1)} \\
+ & \left[\frac{\alpha_{j(n)}}{a_{j(n),j(n)}^n k(j(n),n)} \left(1 - \frac{a_{j(n)+1,j(n)}^n k(j(n),n)}{b_n} \right) \right] \chi_{B(j(n)+1,n+1)} \\
+ & \frac{\alpha_{j(n)+1}}{b_n} \chi_{B(j(n)+1,n+1)}.
\end{aligned}$$

If we let $b_n = a_{j(n)+1,j(n)}^n k(j(n),n)$, then we have the desired outcome.

The significance of this is that when determining $K_0^+(\mathfrak{A})$, we see that since $K_0^+(\mathfrak{A}) = \bigcup_{n \geq 1} \Phi_n(\mathbb{Z}_+^{n+1})$, clearly $K_0^+(\mathfrak{A}) \subset C(X_{min}, G^+)$. Furthermore, for all $U \subset X_{min}$ clopen, $\chi_U \in K_0(\mathfrak{A})$. Therefore, the hypotheses of Theorem 4.4 are satisfied. We then immediately have the following corollaries.

Corollary 5.1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be AF algebras, each which have Bratteli diagrams for which there are unique minimal reductions corresponding to the sets X_{min}^1 and X_{min}^2. If $\mathfrak{A}_1 \cong \mathfrak{A}_2$ then X_{min}^1 is homeomorphic to X_{min}^2.*

Corollary 5.2. *Suppose $\mathfrak{A}$ is an AF algebra with a Bratteli diagram for which there is a unique minimal reduction corresponding to the set X_{min}. Then, every order preserving automorphism ψ of $K_0(\mathfrak{A})$ which takes the order unit onto itself must be of the form $\psi(f) = f \circ \theta^{-1}$ for some homeomorphism θ of X_{min}.*

Remark 7. If X is a 0-dimensional compact metric space then the above arguments show that with $b_n = 1$, for all $n \geq 0$, the methods of this paper provide a direct generalization of the usual way of showing that

$$K_0(C(X)) \cong \left(C(X, \mathbb{Z}), C(X, \mathbb{Z}^+), \chi_X \right).$$

Remark 8. When $\mathfrak{A}$ is such that $m_n = L < \infty$, for all $n \geq 1$, the existence of a unique minimal Bratteli diagram implies the multiplicity

matrices are permutation similar to diagonal matrices. Thus, these corollaries generalize to this situation as well.

We conclude this paper with some applications of Corollary 5.2 which provide information about the automorphism groups of certain AF algebras.

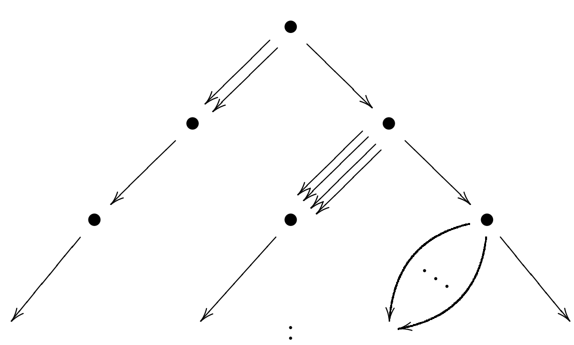
Corollary 5.3. *Suppose $\mathfrak{A}$ is an AF algebra with a Bratteli diagram for which there is a unique minimal reduction corresponding to the set X_{min}. If the only homeomorphism of X_{min} is the identity map, then $Aut(\mathfrak{A}) = \overline{Inn(\mathfrak{A})}$.*

Proof. Each automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A})$ induces an automorphism $\alpha_* \in \mathrm{Aut}(K_0(\mathfrak{A}))$. By Corollary 5.2, $\alpha_*(g) = g \circ \theta^{-1}$ for some homeomorphism θ of X_{min}, where we are representing $K_0(\mathfrak{A})$ as a subset of $C(X_{min}, G)$. By hypothesis, it follows that $\alpha_* = \mathrm{id}_*$, and therefore, by [6, Theorem IV.5.7], $\alpha \in \overline{\mathrm{Inn}(\mathfrak{A})}$. Hence, $\mathrm{Aut}(\mathfrak{A}) = \overline{\mathrm{Inn}(\mathfrak{A})}$. □

Remark 9. In particular, this applies to the UHF algebras, and should be compared with [6, Corollary IV.5.8].

The converse of Corollary 5.3 is in general not true. The following example illustrates this and demonstrates an application of Corollary 5.2 which utilizes the structure of X_{min} to obtain complete information about the automorphism group of $K_0(\mathfrak{A})$.

Example 5.4. Consider the AF algebra $\mathfrak{A}$ with Bratteli diagram



where the multiplicity matrices are such that

$$\overline{A}_{n,n+1} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 2^{n+1} \\ & & & 1 \end{bmatrix} \in M_{n+2,n+1},$$

for all $n \geq 0$. We then complete these matrices in such a way so as to obtain

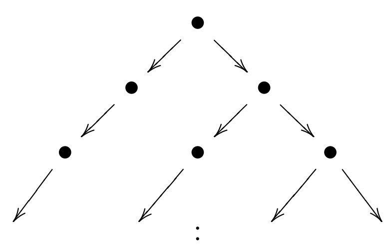
$$A_{n,n+1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 2^{n+1} & \\ & & & 1 & 1 \end{bmatrix} \in M_{n+2}$$

for all $n \geq 0$. Therefore, if we define A_n as in Section 4, one can verify by induction on n that

$$A_n = \begin{bmatrix} 2^{-1} & & & & \\ -2^{-1} & 2^{-2} & & & \\ & -2^{-2} & & & \\ & & \ddots & & \\ & & & 2^{-n} & \\ & & & -2^{-n} & 1 \end{bmatrix}$$

for all $n \geq 1$.

Now, with X_{min} given (uniquely) as the set of all infinite paths in the graph



we have, for any $(\alpha_1, \ldots, \alpha_{n+1})^T \in \mathbb{Z}^{n+1}$, that $\Phi_n(\alpha_1, \ldots, \alpha_{n+1})^T$ is equal to

$$\alpha_1 2^{-1} \chi_{B(1,0)} + \sum_{l=1}^{n-1} (\alpha_{l+1} 2^{-(l+1)} - \alpha_l 2^{-l}) \chi_{B(l+1,l)} + (\alpha_{n+1} - \alpha_n 2^{-n}) \chi_{B(n+1,n)},$$

which, due to the structure of X_{min}, can be written as

$$\left(\sum_{l=1}^{n} \alpha_l 2^{-l} \chi_{B(l,n)} \right) + \alpha_{n+1} \chi_{B(n+1,n)}.$$

Now, by Corollary 5.2, every order preserving automorphism ψ of $K_0(\mathfrak{A})$ must be of the form $\psi(f) = f \circ \theta^{-1}$ for some homeomorphism θ of X_{min}, where $f \in K_0(\mathfrak{A}) \subset C(X_{min}, G)$. The set X_{min} is easily seen

to be homeomorphic to

$$\overline{\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}},$$

and thus, any homeomorphism θ of X_{min} must fix the point corresponding to 0. It follows that if θ is a non-identity homeomorphism of X_{min}, then there exists $n \geq 1$ and $1 \leq l_1 < l_2 \leq n$ such that

$$\theta(B(l_1, n)) = B(l_2, n),$$

where here, $B(l_1, n)$ and $B(l_2, n)$ are singletons. But then, for the function $f = 2^{-l_1}\chi_{B(l_1,n)} + 2^{-l_2}\chi_{B(l_2,n)} \in K_0(\mathfrak{A})$, we see that

$$\psi(f) = f \circ \theta^{-1} = 2^{-l_1}\chi_{B(l_2,n)} + 2^{-l_2}\chi_{B(l_1,n)} \notin K_0(\mathfrak{A}).$$

Hence, we conclude that the only order preserving automorphism of $K_0(\mathfrak{A})$ is the identity map.

Consequently, the converse of Corollary 5.3 is not true. After all, we have just shown that the only order preserving automorphism of $K_0(\mathfrak{A})$ is the identity map. So again, by [6, Theorem IV.5.7], $Aut(\mathfrak{A}) = \overline{Inn(\mathfrak{A})}$. However, X_{min} clearly has many nontrivial homeomorphisms.

Corollary 5.5. *If X is a 0-dimensional compact metric space then $Aut(C(X)) = \{\widehat{\theta} : \theta$ is a homeomorphism of $X\}$ where $\widehat{\theta}(f) = f \circ \theta^{-1}$.*

Proof. Clearly, if θ is a homeomorphism on X, then $\widehat{\theta}$ is an automorphism of $C(X)$. Thus, we must show all automorphisms are of this form.

Let $\alpha \in \mathrm{Aut}(C(X))$. Then, α induces an automorphism α_* of $K_0(C(X))$, which by Corollary 5.2 must be of the form $\alpha_*(f) = f \circ \theta^{-1}$ for some homeomorphism θ of X. But then, for all $f \in K_0(\mathfrak{A})$,

$$\widehat{\theta}_*^{-1} \circ \alpha_*(f) = \widehat{\theta}_*^{-1}\left(f \circ \theta^{-1}\right) = f \circ \theta^{-1} \circ \theta = f.$$

Hence, $\widehat{\theta}_*^{-1} \circ \alpha_* = \mathrm{id}_*$.

By [6, Theorem IV.5.7], it follows that $\widehat{\theta}^{-1} \circ \alpha$ is an approximately inner automorphism of $C(X)$, which of course means $\widehat{\theta}^{-1} \circ \alpha = \mathrm{id}$. Therefore, $\alpha = \widehat{\theta}$.

□

Possibly a more interesting observation than these last two results is that Corollary 5.2 in some sense generalizes Corollary 5.5 to a larger class of algebras. One surprising aspect of this is that the correct set of homeomorphisms is not on the spectrum of $\mathfrak{D} \subset \mathfrak{A}$, but rather on the (often) smaller set X_{min}.

Since the results of this section are for AF algebras with Bratteli diagrams that have unique reductions to X_{min}, we state the following result which provides a characterization of these algebras.

Theorem 5.6. *Suppose there exists a way to complete the matrices* $\{\overline{A}_{n,n+1}\}_{n=0}^{\infty}$ *such that for all* $n \geq 1$,

$$\Phi_n(\alpha_1, \ldots, \alpha_{n+1})^T = \sum_{i=1}^{n+1} \frac{\alpha_i}{k(i,n)} \chi_{B(i,n)},$$

where $k(i,n) \in \mathbb{Z}^+$ *for all* $1 \leq i \leq n+1$. *Then, any two choices for the set* X_{min} *must be homeomorphic.*

Proof. In this case, for any X_{min} associated to $\mathfrak{A}$, we have $K_0^+(\mathfrak{A}) = K_0(\mathfrak{A}) \cap C(X_{min}, G^+)$ and for all $U \subset X_{min}$, clopen, the function χ_U is an element of $K_0(\mathfrak{A})$. Suppose X_{min}^1 and X_{min}^2 are two choices for the set X_{min}. We then know by Elliott's Theorem [9] that there exists an order preserving isomorphism

$$\psi : K_0(\mathfrak{A}) \subset C(X_{min}^1, G_1) \to K_0(\mathfrak{A}) \subset C(X_{min}^2, G_2)$$

which takes order unit to order unit. By Theorem 4.4, it follows that X_{min}^1 and X_{min}^2 are homeomorphic.

□

References

1. Bruce Blackadar, *K-theory for operator algebras*, 2nd ed., Mathematical Sciences Research Institute, vol. 5, Cambridge University Press, 1998.
2. O. Bratteli, P. E. T. Jorgensen, K. H. Kim, and F. Roush, *Non-stationarity of isomorphism between AF algebras defined by stationary Bratteli diagrams*, Ergodic Theory Dynam. Systems **20** (2000), 1639–1656.
3. ———, *Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups*, Ergodic Theory Dynam. Systems **21** (2001), 1625–1655.
4. ———, *Computation of isomorphism invariants for stationary dimension groups*, Ergodic Theory Dynam. Systems **22** (2002), 99–127.
5. Ola Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.
6. Kenneth R. Davidson, *C*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, Rhode Island, 1996.
7. Allan P. Donsig and Alan Hopenwasser, *Analytic partial crossed products*, preprint; arXiv:math.OA/0305337.
8. F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams, and dimension groups*, Ergodic Theory Dynam. Systems **19** (1999), no. 4, 953–993.
9. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.

10. Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.
11. Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), no. 6, 827–864.
12. Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge-New York-Melbourne, 1985.
13. A. J. Lazar and D. C. Taylor, *Approximately finite-dimensional C^*-algebras and Bratteli diagrams*, Trans. Amer. Math. Soc. **259** (1980), no. 2, 599–619.
14. Justin R. Peters and Ryan J. Zerr, *Partial dynamical systems and AF C^*-algebras*, Houston J. Math. (to appear).
15. Stephen C. Power, *Limit algebras: An introduction to subalgebras of C^*-algebras*, Pitman Research Notes in Mathematics, vol. 278, Longman Scientific and Technical, Harlow, Essex, England, 1992.
16. Ian F. Putnam, *The C^*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math. **136** (1989), no. 2, 329–353.

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